Geometric Modeling

Assignment sheet #5

"Basis Functions"

(due June 4th 2012 before the lecture)

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Exercise 1 (Curvature): [2 points]

Given the surface $f(x,y) = 3 + xy$, what is the curvature $\kappa(\alpha)$ at point $\binom{0}{0}$ in direction α (use polar coordinates)? In which direction is it minimal / maximal?

Exercise 2 (Interpolation with Basis Functions): [3+3+1 points]

Let $\{(x_i, y_i) | i = 1 \dots n\}$ be a set of given points. To interpolate them a polynomial of degree n-1 is necessary in general. Derive a formula $L(x)$ for such a function which passes exactly through all the given points.

a. Develop polynomial basis functions $L_k(x)$ of degree n-1 with the following property:

$$L_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad k = 1 \dots n$$

(*Hint:* How does the function $\frac{(x-a)}{(b-a)}$ behave?)

b. Use the basis functions to express the interpolating polynomial $L(x)$.
(*Hint*: sum up the correctly weighted y-entries).

c. Test your formula using the data points sampled from the one dimensional function $f(x) = \sqrt{x}$:

$$f(1) = 1 \quad f(4) = 2 \quad f(16) = 4$$

Use a plotting tool to display this function and the "original curve" $f(x)$ in the same coordinate system and provide a printout in your solution. Also sketch the error function $e(x) = |f(x) - L(x)|$.

Exercise 3 (Bernstein Polynomials): [1+1+1 points]

Prove the following identities for Bernstein Polynomials:

$$\forall x: \sum_{i=0}^{d} b_i^d(x) = 1, \qquad x^n = \sum_{i=k}^{n} \frac{\binom{i}{k}}{\binom{n}{k}} \mathrm{B}_i^n(x), \qquad \frac{d}{dx}\mathrm{B}_i^n(x) = n\left(\mathrm{B}_{i-1}^{n-1}(x) - \mathrm{B}_i^{n-1}(x)\right)$$

Exercise 4 (Compactly Supported Basis Functions): **[5+1+2 points]**

a. We want to interpolate n regularly spaced points (x_i, y_i), $i = 1 \dots n-1$ with $x_i = ih$ sampled from a 1D function $f(x) = y$. To avoid oscillation artifacts, we want to use *piecewise* polynomial basis functions $b_i(x)$ which have the following properties:

$$b_i(x) = \begin{cases} 1 & x = ih \\ 0 & x \leq (i-1)h \\ 0 & x \geq (i+1)h \end{cases} \qquad b_i \in \mathbb{C}^1, i = 1 \dots n-2$$

Derive basis functions b_i which satisfy the properties.

b. Use a plotting tool to draw the function you get by interpolating the sample points

$$s(0) = 1 \quad , \quad s(1) = 2 \quad , \quad s(2) = 3 \quad , \quad s(3) = 4$$

on the interval $[0,3]$ using the basis functions derived in (a) with $h = 1$.

c. How do you need to change the basis functions b_i to make them reproduce constant functions $f(x) = c$ faithfully not only on the sample points but on the whole interval? Write down the new basis functions.